On Maslov Conjecture about Square Root Type Singular Solutions of the Shallow Water Equations*

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1 Introduction

About twenty years ago, V. P. Maslov [1] put forward the idea that numerous quasilinear hyperbolic systems have only finite number of singular solution in general position. These solutions are shock waves, "infinitely narrow" solitons and point singularities of the type of the square root of a quadratic form. He has also stated conjecture that such solutions for shallow water equation can describe the dynamics of mesoscale vortices in the atmosphere, and trajectories of singularities correspond to trajectories of these vortices. Interesting integrability properties of such solutions were found in works [2, 3, 4, 5], where the shallow water equations on the β-plane with variable Coriolis force [6, 7] were considered :

$$\frac{\partial \eta}{\partial t} + \langle \nabla, \eta \mathbf{u} \rangle = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} - \omega \mathbf{T} \mathbf{u} + \nabla \eta = 0. \tag{1}$$

Here $x = (x_1, x_2) \in \mathbf{R}^2$, and the unknowns are the two-dimensional vector $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, and the function η that is the geopotential of the atmosphere or the free surface elevation in the surface waves theory, $\mathbf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $\omega = \tilde{\omega} + \beta x_2$ is the doubled Coriolis frequency on the β-plane, and $\tilde{\omega}$, β are some physical constants. The angle brackets stand for the inner product.

On the some time interval $t \in [0, T]$, the mentioned singular solutions have the following form:

$$\begin{pmatrix} \mathbf{u}(x,t) \\ \eta(x,t) \end{pmatrix} = \begin{pmatrix} u(x,t) \\ \rho(x,t) \end{pmatrix} + \begin{pmatrix} U(x,t) \\ R(x,t) \end{pmatrix} F(S(x,t)), \tag{2}$$

where $F(\tau) = \sqrt{\tau}$. Here the "background" $u = (u_1, u_2)$, ρ and the "vector amplitude" $U = (U_1, U_2), R$ are smooth (vector) functions. The phase function S is a smooth non-negative function vanishing only on the trajectory $x = X(t)$ of the singularity and satisfies additionally the following condition: **Condition (i)** *The matrix* $\mathbf{H}(t) = (\partial^2 S/\partial x_i \partial x_j)(X(t), t)$ *is strictly positive and has distinct eigenvalues.*

Some arguments in favor of the importance of solutions of the form (2) are given in [1, 2, 3, 4, 5, 8, 9]; these solutions are elements of some algebra of singular solutions of quasilinear hyperbolic systems. The solutions of the form (2) are selected from a much wider class of singular

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solutions by the general position condition (i). This class consists of solutions of the form (2) with functions F satisfying the following conditions: **Condition (ii)** a. *$F(\tau)$ is a continuous functions as $\tau \geq 0$, $F(0) = 0$;* b. *$F(\tau)$ is smooth for $\tau > 0$, $\lim_{\tau \to +0} F'(\tau) = \infty$.*

Obviously, the function F in (2) is not determined uniquely by the condition (ii): we can multiply it on any non-vanishing smooth function and add any smooth function vanishing on the trajectory $X(t)$. Moreover, the function S can be also multiplied by non-vanishing smooth function.

As was noted, it was announced in [1], that for numerous hyperbolic systems the condition (i) "kills" (modulo mentioned possible changes in (2)) all the solutions (2) excepting $F = \sqrt{\tau}$ in the sense that if $F \neq \sqrt{\tau}$, then all the x-derivatives of U and R are zero on the trajectory $X(t)$. The proof of this conjecture was given first by V. N. Zhikharev [8], but his arguments were not complete even they contained many elegant constructions. It is impossible to rectify his proof, because it needs really considering of the problem from the very beginning.

Our aim is to give a complete proof of the Maslov conjecture for equations (1). There is no enough place in this paper for the whole proof, so here we only give the general scheme, accent the principal moments and the ideas, formulate basic main and auxiliary statements, and illustrate proofs for some of them. The complete proof will appear in [10]. Our approach is based on algebraic construction rather than to them related to differential equations, and it seems, that the similar considerations are valid for much wider class of hyperbolic quasilinear system.

Theorem *Suppose that the functions specifying the solution* (2) *satisfy the conditions* (i) *and* (ii)*, and some x-derivatives of U and R are non-zero. Then without loss of generality we can assume $F = \sqrt{\tau}$.*

2 Auxiliary constructions

First of all, we pass to the moving coordinate system (x', t) with $x' = x - X(t)$, then system (1) is written as (we omit the prime on the new x variable and put $V = \dot{x}$, $v = \mathbf{u} - V$)

$$\eta_t + \langle \nabla, \eta v \rangle = 0, \quad v_t + \langle v, \nabla \rangle v + \nabla \eta + \dot{V} - \omega \mathbf{T}(v + V) = 0.$$

Substitute now function (2) into this system, then we obtain the system

$$F'(\tau) A + F(\tau) B + F(\tau) F'(\tau) C + F^2(\tau) D + E = 0. \tag{3}$$

Here

$$A = \begin{pmatrix} \Lambda R + \rho \langle U, P \rangle \\ RP + \Lambda U \end{pmatrix}, \; B = \begin{pmatrix} R_t + \langle \nabla, \rho U + Ru \rangle \\ \Gamma(U) + \langle U, \nabla \rangle u + \nabla R - w \mathbf{T} U \end{pmatrix}, \; C = \begin{pmatrix} 2R \langle U, P \rangle, \\ \langle U, P \rangle U \end{pmatrix},$$
$$D = \begin{pmatrix} \langle \nabla, RU \rangle \\ \dot{U} \end{pmatrix}, \quad E = \begin{pmatrix} \rho_t + \langle \nabla, \rho u \rangle \\ \Gamma(u) + \nabla \rho + \dot{V} - \omega \mathbf{T}(u + V). \end{pmatrix},$$

where $\tau = S(x,t)$, $P = \nabla S$, $\Lambda = S_t + \langle u, P \rangle$, $\dot{U} = \langle U, \nabla \rangle U$, $\Gamma = \partial / \partial t + \langle u, \nabla \rangle$. Since the function S can be represented in the form (see condition (i)) $S = \frac{1}{2} \langle x, \mathbf{H}(t) x \rangle + O(|x|^3)$, by using the Morse lemma [11] there exists a smooth change of variables $x = x(y,t)$ such that $S = y^2$. The vectors A, B, C, D, and E depend smoothly on y. We can also introduce polar coordinates τ, φ, $y_1 = \sqrt{\tau} \cos \varphi$, $\quad y_2 = \sqrt{\tau} \sin \varphi$, and substitute them into system (3), then for each φ we obtain a system of first order differential equations with coefficients depending smoothly on $\sqrt{\tau}$, φ, e.c. $A = A(x(y(\sqrt{\tau}, \varphi)))$. To simplify the notation we do not write all these arguments and keep only dependence of coefficients on x.

We will intensively exploit the fact that F *does not depend on* φ.

Our proof is divided now into three main parts:

- to obtain from (3) more simple equations admitting exact formulas for their solutions,
- to select from these solutions only those satisfying (ii);
- to show inconsistency of system (3) corresponding to all the obtained solutions except $F = \sqrt{\tau}$.

3 Model equations

Since we have a system of three equations for a single function F, one can try to eliminate terms with F' and FF' and to obtain a quadratic equation. Unfortunately, because of degeneracy, it is not always possible, and we have to find another way of simplification for equations (3).

We set $P_\perp = \mathbf{T}P$, $U_\perp = \mathbf{T}U$. Let $G(x,t)$ be a smooth scalar or vector function. Then to G we can assign the Taylor expansion at $x = 0$:

$$G \sim \sum_{n=0}^{\infty} G^{(n)}(x,t),$$

where $G^{(n)}(x,t)$ is a homogeneous polynomial of x of degree n. By $o(|x|^\infty)$ we denote smooth functions vanishing to the infinite order at $x = 0$.

Lemma 1 (On model equations) *Let the condition* (i) *is fulfilled and system* (3) *has a solution F satisfying condition* (ii), *then*

A. *there exist smooth functions α, β, γ, δ of (x,t) such that F satisfies the Riccati equation*

$$\alpha(x,t)F'(\tau) + \beta(x,t)F^2(\tau) + \gamma(x,t)F(\tau) + \delta(x,t) = 0, \tag{4}$$

where x-derivatives of the function α do not vanish at $x = 0$.

B. *F satisfies one of the following three equations:*

- the quadratic equation

$$a(x,t)F^2(\tau) + b(x,t)F(\tau) + c(x,t) = 0, \tag{5}$$

if $\langle U, P_\perp\rangle \neq o(|x|^\infty)$, $\langle \dot{U}, U_\perp\rangle \neq o(|x|^\infty)$, and $\langle U, P\rangle = o(|x|^\infty)$, or $\langle U, P_\perp\rangle = o(|x|^\infty)$,

- the linear differential equation

$$a(x,t)F'(\tau) + b(x,t)F(\tau) + c(x,t) = 0, \tag{6}$$

if $\langle U, P_\perp\rangle \neq o(|x|^\infty)$, $\langle \dot{U}, U_\perp\rangle \neq o(|x|^\infty)$,

- the cubic equation

$$a(x,t)F^3(\tau) + b(x,t)F^2(\tau) + c(x,t)F(\tau) + d(x,t) = 0, \tag{7}$$

if $\langle U, P_\perp\rangle \neq o(|x|^\infty)$, $\langle \dot{U}, U_\perp\rangle \neq o(|x|^\infty)$, $\langle U, P\rangle \neq o(|x|^\infty)$.

Here also a, b, c, d are smooth functions and some x-derivatives of a do not vanish at the point $x = 0$.

C. *if $\psi = o(|x|^\infty)$, then $F^n \partial^m F/\partial\tau^m \psi = o(|x|^\infty)$ for any integer n and integer $m \geq 0$.*

Prior to proving lemma 1, we prove a useful auxiliary assertion.

Lemma 2 *Let a smooth vector function z satisfy $\langle z, P\rangle = 0$, then there exists a smooth function α such that $z = \alpha P_\perp$. It follows from here, that if $z = z^{(k)}$ is k-st-order homogeneous polynomial in x, $z^{(k)} \neq 0$, then the same assertion is valid with z replaced by $z^{(k)}$, P replaced by $P^{(1)}$, and α replaced by $(k-1)$-st-order homogeneous polynomial $\alpha^{(k-1)}$.*

Obviously, it suffices to prove the assertion pertaining to z. We pass from the coordinates x to the Morse coordinates y. Then we have $P = {}^t(\partial x/\partial y)^{-1}\,{}^t(y_1, y_2)$. Consequently, $\langle(\partial x/\partial y)^{-1} w, y\rangle = 0$. We set $w = (\partial x/\partial y)^{-1} z$ and represent the components w_1 and w_2 of the vector w in the form $w_1 = w_1^0(y_1, t) + y_2 \tilde{w}_1(y, t)$ and $w_2 = w_2^0(y_2, t) + y_1 \tilde{w}_2(y, t)$, where w_i^0 and $\tilde{w}_i$ are smooth functions. Then the orthogonality condition for z and P acquires the form $y_1 w_1^0(y_1, t) + y_2 w_2^0(y_2, t) + y_1 y_2(\tilde{w}_1 + \tilde{w}_2) = 0$. In this equation, we in turn set $y_1 = 0$ and $y_2 = 0$, thus obtaining $w_1^0 = w_2^0 = 0$. Next, we divide it by $y_1 y_2$ and obtain $\tilde{w}_1 = -\tilde{w}_2 \equiv \tilde{g}(y, t)$ and $(\partial x/\partial y)^{-1} z = gTY$. Now note that $T^t Q = \det Q Q^{-1} T$ for any non-degenerate 2×2 matrix Q. Therefore,

$$z = g\Big(\frac{\partial x}{\partial y}\Big) T y = g \det\Big(\frac{\partial x}{\partial y}\Big) T^t \Big(\frac{\partial x}{\partial y}\Big)^{-1} y = g\Big(\frac{\partial x}{\partial y}\Big) P_\perp.$$

The proof of the lemma 2 is complete.

To prove the lemma 1, we use mostly pure algebraic procedures. Two cases are studied separately:

$$(a) \quad \langle U, P_\perp\rangle \neq o(|x|^\infty), \quad (b) \quad \langle U, P_\perp\rangle = o(|x|^\infty).$$

Study first the case (a). Let us transforms the system (3). The coefficients of the transformed systems will be indicated by primes; we do not write system themselves, but only their coefficients, which are again referred to (3).

Let us multiply the first equation of (3) by U in the sense of the inner product, and the second and the third equation by $2R$; then we subtract the first equation from the second and the third ones; this leads to the equations with coefficients

$$A' = \begin{pmatrix} \Lambda R + \rho\langle U, R\rangle \\ 2R^2 P + (\Lambda R - \rho\langle U, P\rangle) U \end{pmatrix}, \; C' = \begin{pmatrix} 2R\langle U, P\rangle \\ 0 \end{pmatrix}, \; D' = \begin{pmatrix} \langle \nabla, RU\rangle \\ 2R\dot{U} - \langle \nabla, RU\rangle U \end{pmatrix}.$$

In the primed system, we multiply the second and the third equations by $U_\perp$ and $P_\perp$, this results in system (3) with coefficients

$$A'' = \begin{pmatrix} \Lambda R + \rho\langle U, P\rangle \\ 2R^2\langle U_\perp, P\rangle \\ (\rho\langle U, P\rangle - \Lambda R)\langle U, P_\perp\rangle \end{pmatrix}, \quad C'' = \begin{pmatrix} 2R\langle U, P\rangle \\ 0 \\ 0 \end{pmatrix},$$
$$D'' = \begin{pmatrix} \langle \nabla, RU\rangle \\ 2R\langle \dot{U}, U_\perp\rangle \\ 2R\langle \dot{U}, P_\perp\rangle - \langle \nabla, RU\rangle\langle U, P_\perp\rangle \end{pmatrix}.$$

The second equation in the obtained system is the requested Riccati equation (4). The conclusion C. of the lemma follows directly from the Riccati equation.

Consider now the following sub-cases:

$$(a.1) \quad \langle \dot{U}, U_\perp\rangle = o(|x|^\infty), \quad (a.2) \quad \langle \dot{U}, U_\perp\rangle \neq o(|x|^\infty).$$

In case $(a.1)$ the term $\langle \dot{U}, U_\perp\rangle F^2$ in the obtained Riccati equation can be included into the coefficient E_2'', and we arrive at the equation (6).

Now proceed with the case $(a.2)$. Suppose first $(a.2.1)$ $\langle U, P\rangle \neq o(|x|^\infty)$. Let us express F' through F^2 and F from the Riccati expression and substitute this expression into the first equation. This results in the cubic equation (7) with non-degenerate coefficient at F^3. If $(a.2.2)$ $\langle U, P\rangle = o(|x|^\infty)$, we multiply the first equation of the double primed system by $\langle U_\perp, P\rangle$ and add it to the third equation of the same system; as a result the coefficients of F' and FF' in the obtained equation are $o(|x|^\infty)$, and these terms can be included into the corresponding E-coefficient. The coefficient of F^2 is $D_3''' = 2(R\langle \dot{U}, P_\perp\rangle - \langle \nabla, RU\rangle\langle U, P_\perp\rangle)$. Under lemma 2, U can be represented in the form $U = \alpha P_\perp + \beta P$, where α, β are smooth function, $\beta = o(|x|^\infty)$, and some derivatives of α do not vanish at $x = 0$. Some calculations reduce expression for D_3''' to the form $2\alpha^2\langle P, \mathbf{Q}P\rangle + \zeta$, where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} -R\dfrac{\partial^2 S}{\partial x_1 \partial x_2} - \langle P_\perp, \nabla R\rangle & R\dfrac{\partial^2 S}{\partial x_1^2} \\ R\dfrac{\partial^2 S}{\partial x_2^2} & -R\dfrac{\partial^2 S}{\partial x_2 \partial x_1} - \langle P_\perp, \nabla R\rangle \end{pmatrix},$$

and $\zeta = o(|x|^\infty)$. Suppose now that $D_3''' = o(|x|^\infty)$, then by virtue of Lemma 2 we obtain $\mathbf{Q}P = \gamma P + \delta P_\perp$, where γ, δ are smooth functions, and $\gamma = o(|x|^\infty)$. This means, in particular, that up to $o(|x|^\infty)$ we have $S''_{x_1x_1} = -S''_{x_2x_2}$. But in this case the matrix $\mathbf{H}$ (see condition (i)) is not positive definite. This contradiction finishes the proof for case (a).

The consideration of the case (b) is based on similar ideas, but needs more sophisticated calculation.

4 Possible solutions to the model equations

Our previous consideration shows that coefficients of equations (3) are smooth functions of $\sqrt{\tau}$ only. Nevertheless, using the assumption about the existence of solutions satisfying (ii), we show that we can really suppose them smooth on τ.

The following assertion plays an important role in all our considerations:

Lemma 3 *Let u, v be smooth functions of y and some derivatives of v at $y = 0$ do not vanish. If a function Φ satisfies $\Phi(\tau) = u(y_1, y_2)/v(y_1, y_2)$, then $\Phi(\tau) = \tau^n\Psi(\tau)$, where Ψ is a smooth function, n is an integer number.*

For the proof, we set $y_1 = \sqrt{\tau}\cos\varphi$,$y_2 = \sqrt{\tau}\sin\varphi$, $\alpha(\tau,\varphi) = u(\sqrt{\tau}\cos\varphi, \sqrt{\tau}\sin\varphi)$, $\beta(\tau,\varphi)$ $=v(\sqrt{\tau}\cos\varphi, \sqrt{\tau}\sin\varphi)$, then α and β are smooth functions of $\sqrt{\tau}$, and $f(\tau) = \frac{\alpha(\tau,\varphi)}{\beta(\tau,\varphi)}$. Let us extract the leading terms of α and β:

$$\alpha = \alpha_m(\varphi)\tau^{m/2} + a_m(\tau,\varphi)\tau^{m/2}, \quad \beta = \beta_k(\varphi)\tau^{k/2} + b_k(\tau,\varphi)\tau^{k/2}, \quad a_m(0,\varphi),\, b_k(0,\varphi) = 0. \quad (8)$$

If $p = m - k$ is odd, then at least in some sector the leading term of f is $\frac{\alpha_m(\varphi)}{\beta_k(\varphi)}\tau^{p/2}$. Note that α_m and β_k can be represented as homogeneous polynomials of order m and k, respectively, in $\cos\varphi$ and $\sin\varphi$, and one can readily see that the expression (8) in this case (odd p) depends on φ. Therefore, the number p must be even. Then we can write

$$\Psi = \frac{W}{Z}, \text{ where } \Psi(\tau) = \frac{f(\tau)}{\tau^n}, \quad W(\tau,\varphi) = \alpha(\tau,\varphi)\tau^{-m/2}, \quad Z(\tau,\varphi) = \beta(\tau,\varphi)\tau^{-k/2}. \quad (9)$$

It is obvious that $W(0,\varphi), Z(0,\varphi) \neq 0$, and therefore, Ψ is a smooth function of $\sqrt{\tau}$ and satisfies $\Psi(0) \neq 0$. For brevity, we set $\xi = \sqrt{\tau}$. Now for each N we can write $\Psi = \sum_{l=0}^N \Psi_l\xi^l + o(\xi^N)$, $W = \sum_{l=0}^N W_l(\varphi)\xi^l + o(\xi^N)$, $Z = \sum_{l=0}^N Z_l(\varphi)\xi^l + o(\xi^N)$. Let us substitute these sums into (9)

and match the coefficients of ξ^l; then for $l \leq N$ we obtain $W_l = \sum_{j=0}^{l} \Psi_j B_{l-j}$. By taking N sufficiently large, we can obtain the last expression for every l. These equations for odd l read as follows:

$$W_1 = \Psi_0 Z_1 + \Psi_1 Z_0, \quad W_3 = \Psi_0 Z_3 + \Psi_1 Z_2 + \Psi_2 Z_1 + \Psi_3 Z_0,$$
$$W_5 = \Psi_0 Z_5 + \Psi_1 Z_4 + \Psi_2 Z_3 + \Psi_3 Z_2 + \Psi_4 Z_1 + \Psi_5 Z_0, \ldots$$

From the first of these equations, we obtain $\Psi_1 = 0$ (indeed, $W_1 - \Psi_0 Z_1$ is a polynomial of odd order in $\cos\phi$ and $\sin\phi$ and cannot be equal to the even-order polynomial $\Psi_1 Z_0$). By substituting this into the second equation, we obtain $\Psi_3 = 0$, and so on. Finally, we see that $\Psi_l = 0$ for all odd l. Recall that $\Psi_l = d^l\Psi/d\xi^l(0)$, and there obviously exists a smooth function Φ such that $\Psi(\xi) = \Phi(\tau)$. Now we only need to recall that $\Psi = \tau^{-n} f$.

In what follows we use differentiation by φ: $\frac{\partial z}{\partial \varphi} = -y_2 \frac{\partial z}{\partial y_1} + y_1 \frac{\partial z}{\partial y_2}$. The important fact is that if z is also smooth, then z_φ is also smooth.

Lemma 4 *Let u, v be smooth functions of y and some derivatives of v at the point $y = 0$ do not vanish. If all the derivatives of the function $u'_\varphi v - u v'_\varphi$ vanish at $y = 0$, then at least in some sector $\varphi_1 \leq \varphi \leq \varphi_2$ we have $u = \tau^n \alpha v$, where n is an integer number, α is a smooth function of τ*

Let us consider quadratic equation (5) and linear differential equation (6). Divide both them by a and differentiate by φ, then in both cases we obtain a linear equation $uF + v = 0$, where $u = b'_\varphi a - b a'_\varphi$, $v = c'_\varphi - c a'_\varphi$. If u has non-vanishing derivatives at $y = 0$, then Lemma 3 and condition (iia) imply smoothness F, and, therefore, F does not satisfy the condition (iib). Therefore, all the derivatives of both u and v vanish at $y = 0$. By applying Lemma 4 we obtain $b = \tau^{-n}\beta a$, $c = \tau^{-n}\gamma a$, where n is a non-negative integer number and β, γ are smooth functions of τ. Dividing now equations (5), (6) by a, we obtain the equations of the same type:

$$\tau^n F^2 + \beta F + \gamma = 0, \tag{10}$$
$$\tau^n F' + \beta F + \gamma = 0, \tag{11}$$

but with coefficients depending on τ smoothly.

Now consider the cubic equation (7). Divide it also by a and differentiate by φ, then we obtain a quadratic equation $uF^2 + vF + w = 0$, where $u = b'_\varphi a - b a'_\varphi$, $v = c'_\varphi - c a'_\varphi$, $w = d'_\varphi a - d a'_\varphi$. If u has non-vanishing derivatives at $y = 0$, then we have a quadratic equation, which is already studied. If all the derivatives of u at $y = 0$ vanish, we can include the term uF^2 into w, and obtain a linear equation $vF + W$, $W = w + vF^2$. Again by using Lemma 3 and condition (ii) we show that all the derivatives of both v and W (and, consequently, w) vanish at $y = 0$. We have again $b = \tau^{-n}\beta a$, $c = \tau^{-n}\gamma a$, $d = \tau^{-n}\delta a$, where n is a non-negative integer number and β, γ, δ are smooth functions of τ. Dividing (7) by a and multiplying by τ^n we reduce it to a cubic equation with smooth on τ coefficients:

$$\tau^n F^3 + \beta F^2 + \gamma F + \delta = 0, \quad \alpha = \tau^n. \tag{12}$$

Now we can use exact formulas for solutions of obtained equations.

For **quadratic equation** (10) we have $F = (-\beta \pm \sqrt{\beta^2 - 4\gamma\tau^n})/(2\tau^n)$. If $\beta^2 - 4\gamma\tau^n \neq o(|x|^\infty)$, then condition (ii) immediately gives us $F = \sqrt{\tau}$. If $\psi = \beta^2 - 4\gamma\tau^n = o(|x|^\infty)$, then condition (iia) gives us at least the representation $F = f + \sqrt{\psi}$ with a smooth function f. Let us substitute this representation into Riccati equation (4), than we obtain for $\sqrt{\psi}$ a Riccati

equation with smooth coefficients. This means, that $\sqrt{\psi}$ is a smooth function, and F does not satisfy condition (iib).

Now let us consider **linear equation** (11). Consider different cases.

Suppose that $\beta = o(|y|^\infty)$, then the function $\delta = \beta F + \gamma$ is smooth, $F' = -\delta/\tau^n$, and condition (iia) implies smoothness of F. Let $\gamma = o(|y|^\infty)$, then $\delta = \gamma/F$ is smooth, and we have $\tau^n F'/F + e = 0$, where $e = \beta + \delta$ is a smooth function. Consequently, $\frac{F'(\tau)}{F(\tau)} = -\frac{e(y_1,y_2)}{\tau^n}$, or by Lemma 3, $\frac{F'(\tau)}{F(\tau)} = \tau^m \Phi(\tau)$, where Φ is a smooth function. Then $F(\tau) = \exp \int \tau^m \Phi(\tau) d\tau$. Without loss of generality we can suppose $\Phi(0) \neq 0$. If $m \geq 0$, then $F(0) \neq = 0$. If $m < -1$, then either $F(0) = \infty$ or $F'(0) = 0$. Suppose $m = -1$, then $F(\tau) = \tau^\kappa \Psi(y)$, where $\kappa = \Phi(0)$, Ψ is a smooth function non-vanishing at $y = 0$. The condition (ii) is satisfied if and only if $0 < \kappa < 1$.

Now suppose that both β and γ have non-vanishing derivatives. Let us rewrite our equation in form $F' = AF + B$, where $A = \tau^m \beta_0$, $B = \tau^k \gamma_0$, where m, k are integers, β_0, γ_0 are smooth functions non-vanishing at $y = 0$, then F can be written in a form

$$F(\tau) = \Big(\int B(\tau) \exp \Big(-\int A(\tau) d\tau \Big) d\tau \Big) \exp \Big(\int A(\tau) d\tau \Big).$$

Suppose **(a)** $m \geq 0$, then $\phi_0 = \exp \int \alpha(\tau)\, d\tau$ is smooth and $\phi_0(0) \neq 0$. Also

$$\int B(\tau) \exp \Big(-\int A(\tau)\, d\tau \Big) d\tau = \int \frac{B(\tau)}{\phi_0(\tau)}\, d\tau = \tau^p \psi_1(\tau) + q \log \tau + \psi_2(\tau)$$

for some smooth functions $\psi_1(\tau)$ and $\psi_2(\tau)$, $p \leq 0$, $q \in \mathbf{R}$, and $F(\tau) = q\phi_0(\tau) \log \tau + \tau^p \phi_1(\tau) + \phi_2(\tau)$, where ϕ_1 and ϕ_2 are smooth functions. One can readily see that condition (iia) implies $q = 0$; therefore, F cannot satisfy (iib).

Now suppose **(b)** $m = -1$, then $\int A(\tau)\, d\tau = \kappa \log \tau + \psi(\tau)$, for some $\kappa \neq 0$ and some smooth function ψ, and, consequently, $\phi_0 = \exp \int a(\tau)\, d\tau = \tau^\kappa \phi_1(\tau)$ for some smooth function $\phi_1(\tau)$ with $\phi_1(0) \neq 0$. Suppose first that κ is integer, then

$$\int \frac{B(\tau)}{\phi_0(\tau)}\, d\tau = \tau^p \psi_1(\tau) + Z \log \tau + \psi_2(\tau),$$

where $p \leq 0$, ψ_1 and ψ_2 are smooth functions, $Z \in \mathbf{R}$, and $F(\tau) = \tau^\kappa \Phi_1(\tau) + \tau^l \Phi_2(\tau) \log \tau + \Phi_3(\tau)$, where Φ and Ψ are smooth functions and $\Phi(0) \neq 0$. Condition (ii) permits us to rewrite this in a form $F(\tau) = \tau(\Phi(\tau) \log \tau + \Psi(\tau))$, where Φ and Ψ are smooth functions and $\Phi(0) \neq 0$. If κ is not integer, then $\int B(\tau)/\phi_0(\tau)\, d\tau = \tau^{l-\kappa} \psi(\tau) + C$ for some smooth function ψ and some integer l. We have $F(\tau) = \tau^l \psi(\tau) \phi_1(\tau) + C \phi_1(\tau) \tau^\kappa$. Condition (ii) implies $0 < \kappa < 1$.

Now let **(c)** $m \leq -2$, then we obtain

$$F(0) = C + \int_\sigma^0 (\tau^m \alpha_0(\tau) F(\tau) + \tau^n \beta_0(\tau))\, d\tau. \tag{13}$$

Let us extract leading terms of both summands in the integrand; they are given by $\mu_1(\tau) = \alpha_0(0) \tau^m F(\tau)$ and $\mu_2(\tau) = \beta_0(\tau) \tau^k$. Taking into account condition (ii), we have

$$\lim_{\tau \to 0} \frac{\mu_1(\tau)}{\mu_2(\tau)} = \frac{\alpha_0(0)}{\beta_0(0)} \lim_{\tau \to 0} \frac{F(\tau)}{\tau^{k-m}} = \begin{cases} \infty, & k > m, \\ 0, & k \leq m. \end{cases}$$

Thus, if $k > m$, then the leading term of the integrand in (13) is $\mu_1(\tau)$. Since $1/\tau = o(\mu_1)$ in this case, it follows that the integral in (13) diverges, which means that we have no desired solutions. If $k \leq n$, then the leading term is $\mu_2(\tau)$, and the integral in (13) obviously diverges as well. Therefore our linear differential equation is considered.

Cubic equation (12) by standard substitution $F = z + g(\tau)$, $g = -\beta/(3\tau^n)$ is reduced to the canonical form $z^3 + pz + q = 0$, where $p = -(-\gamma/\tau^n + \beta/(3\tau^n))$, $q = \delta/\tau^n - 2\beta^3/(3\tau^n)^2 - \beta\gamma/(3\tau^2 n)$. We use the Cardano formula

$$z = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3},$$

where the branches of a cubic root are chosen in such a way that the product of two summands is equal to $-p/3$. Further analysis is based on the comparing of the orders of p and q in the neighborhood of the point $y = 0$. This analysis is quite simple, although requires certain calculations. We omit it here, and note only, that in the case $\frac{q^2}{4} + \frac{p^3}{27} = o(|x|^\infty)$ we have to use the Riccati equation (4) like it was done for the quadratic equation.

The following assertion summarizes all our considerations.

Lemma 5 *Without loss of generality, the model equations* (5), (6), (7) *can have only the following solutions satisfying* (ii)*:*

$$\begin{array}{ll} \text{(F1)}\ F = \tau^\kappa, & \text{(F2)}\ F = \tau \log \tau, \\ \text{(F3)}\ F = \tau^{1/3} + \sigma\tau^{2/3+n}, & \text{(F4)}\ F = \tau^{2/3} + \sigma\tau^{4/3+n}, \end{array}$$

where $0 < \kappa < 1$*,* $\sigma = \pm 1$ *or* 0*,* n *is an integer number. The quadratic equation* (5) *has only solution of the type* (F1) *with* $\kappa = 1/2$*, the linear differential equation* (6) *has solutions of the types* (F1) *and* (F2)*, the cubic equation* (7) *has solutions* (F1) *with* $\kappa = 1/2$*,* (F3) *and* (F4).

5 Original system of equations and the singularities of selected types

Turn back to system (3), and substitute obtained singular solutions into the system.

Lemma 6 *Suppose* $\kappa \neq 1/2$*, then up to* $o(|x|^\infty)$ *the following relations hold:*

$$\text{(F1)} \quad : \quad \kappa A + BS = 0, \quad \kappa C + DS = 0, \quad E = 0; \tag{14}$$

$$\text{(F2)} \quad : \quad A + BS + CS = 0, \quad C + DS = 0, \quad A + E = 0; \tag{15}$$

$$\text{(F3)} \quad : \quad \left\{ \begin{array}{l} A + 3BS + (2+3n)CS^{2n+1} + 3DS^{2n+2} = 0, \\ C + 3DS + (2+3n)\sigma AS^n + 3\sigma BS^{n+1} = 0, \\ E + (1+n)\sigma CS^n + 2\sigma DS^{n+1} = 0; \end{array} \right. \tag{16}$$

$$\text{(F4)} \quad : \quad \left\{ \begin{array}{l} 2A + 3BS + (4+3n)CS^{2n+2} + 3DS^{2n+3} = 0, \\ 2C + DS + (4+3n)\sigma AS^n + 3\sigma BS^{n+1} = 0, \\ E + (2+n)\sigma CS^{n+1} + 2\sigma DS^{n+2} = 0. \end{array} \right. \tag{17}$$

Each of these systems is consistent only if all the derivatives of U *and* R *vanish on the trajectory* $X(t)$*. In other words, singular solutions of the types* (F1)–(F4) *with* $\kappa \neq 1/2$ *do not exist.*

The proof of equalities (14)–(17) is obtained in a direct way. The proof of inconsistency is more delicate.

Decompose all the functions into Taylor series and substitute these series into systems (14)–(17), and equate coefficients at the least powers of x. The first three equations have the same form

$$A^{(1)} = \left(\begin{array}{c} \langle u^{(0)}, P^{(1)} \rangle R^{(0)} + \rho^{(0)} \langle U^{(0)}, P^{(1)} \rangle \\ R^{(0)} P^{(1)} + \langle u^{(0)}, P^{(1)} \rangle U^{(0)} \end{array} \right) = 0.$$

Lemma 7 $U^{(0)} = 0,\ R^{(0)} = 0.$

For the proof, let us multiply the first equation by $\langle u^{(0)}, P^{(1)}\rangle$ and subtract the second equation multiplied by $P^{(1)}$ in the sense of the inner product and then by $\rho^{(0)}$. We obtain

$$(\langle u^{(0)}, P^{(1)}\rangle^2 - \rho^{(0)}(P^{(1)})^2)R^{(0)} = 0.$$

Since $(P^{(1)})^2$ is a quadratic form, $\rho^{(0)} \neq 0$, and $\langle u^{(0)}, P^{(1)}\rangle$ is a linear form, it follows that the first factor in the resulting equation does not vanish. Consequently, $R^{(0)} = 0$. Then it readily follows from the first equation that $\langle U^{(0)}, P^{(1)}\rangle = 0$, and so $U^{(0)} = 0$. Proof is complete.

Now suppose that for some $k \geq 1$ we have already $U_j = 0$, $R_j = 0$, $j < k$. Let us write out the least-order terms in the first vector equations in systems (14)–(17). They all have the same form:

$$\begin{aligned} R^{(k)}\langle u^{(0)}, P^{(1)}\rangle + \rho^{(0)}\langle U^{(k)}, P^{(1)}\rangle + \nu S^{(2)}(\rho^{(0)}\langle \nabla, U^{(k)}\rangle + \langle u^{(0)}, \nabla\rangle R^{(k)}) &= 0, \quad (18)\\ R^{(k)}P^{(1)} + \langle u^{(0)}, P^{(1)}\rangle U^{(k)} + \nu S^{(2)}(\nabla R^{(k)} + \langle u^{(0)}, \nabla\rangle U^{(k)}) &= 0, \quad (19)\end{aligned}$$

where ν is a number depending on the system.

Lemma 8 *Under condition* $u^{(0)} \neq 0$ *we have* $U^{(k)} = 0,\ R^{(k)} = 0.$

This assertion for system (18) - (19) was proved in [8, 10]. We omit it.

Lemma 9 *Suppose that* $R^{(j)} = 0$ *and* $U^{(j)} = 0$ *for* $j < k$, *where* $k \geq 1$. *Then* $R^{(k)} = 0$.

Since $u^{(0)} = 0$, equation (19) becomes $R^{(k}P^{(1)} + \nu S^{(2)}\nabla R^{(k)} = 0$. Let us multiply it by x. By the Euler identity $\langle x, \nabla R^{(k)}\rangle = kR^{(k)}$, and $\langle P^{(1)}, x\rangle = 2S^{(2)}$. Hence $R^{(k)}(\nu k + 2) = 0$, which implies $R^{(k)} = 0$. The proof is complete.

Thus, to finish the proof of lemma 6 we have to show that $U^{(k)} = 0$ under the following assumption: $u^{(0)} = 0$ and $R^{(j+1)} = 0$, $U^{(j)} = 0$ for $j < k$, where $k \geq 1$

Let us write out the first vector equations for the least powers of x in (14)–(17). They have the same form:

$$\begin{aligned} \langle U^{(k)}, P^{(1)}\rangle + \nu S^{(2)}\langle \nabla, U^{(k)}\rangle &= 0, \quad (20)\\ R^{(k+1)}P^{(1)} + \Lambda^{(2)}U^{(k)} + \nu S^{(2)}(U_t^{(k)} + \langle u^{(1)}, \nabla\rangle U^{(k)} + & \quad (21)\\ \langle U^{(k)}, \nabla\rangle u^{(1)} + \nabla R^{(k+1)} - \omega^{(0)}TU^{(k)}) &= 0.\end{aligned}$$

Now we distinguish between equations (14), (15) and equations (16), (17). We start from equations (14) and (15). In the following, we need the second vector equations in systems (14)–(15). We equate the coefficient of the least power in the expansion in powers of x with zero. This results in the equations

$$\begin{aligned} R^{(k+1)}\langle U^{(k)}, P^{(1)}\rangle + \nu S^{(2)}\langle \nabla, R^{(k+1)}U^{(k)}\rangle &= 0 \quad (22)\\ \langle U^{(k)}, P^{(1)}\rangle U^{(k)} + \nu S^{(2)}\langle U^{(k)}, \nabla\rangle U^{(k)} &= 0. \quad (23)\end{aligned}$$

Lemma 10 *Equations* (20), (21) *and* (23) *are compatible if and only if* $U^{(k)} = 0$.

Let us multiply the (21) by $P^{(1)}$ and use the expression for $\langle U^{(k)}, P^{(1)}\rangle$. We obtain that $R^{(k+1)}$ is divided by $S^{(2)}$: $R^{(k+1)} = \widetilde{R}^{(k-1)}S^{(2)}$, Now we note that if $U^{(k)} = S^{(2)}\widetilde{U}^{(k-2)}$, then the substitution $U^{(k)}$, $R^{(k+1)}$ into (20), (21), (23) gives the same system for $\widetilde{U}^{(k-2)}$, $\widetilde{R}^{(k-1)}$ with

$\widetilde{\nu} = \nu/(1+\nu)$ instead of ν. Hence without loss of generality we can assume that $U^{(k)}$ is not divisible by $S^{(2)}$ (otherwise, we arrive at the original system with $k < 2$ by finitely many steps).

From equations (20), the relation $2S^{(2)} = (P^{(1)}, x)$, and Lemma 2 we obtain

$$U^{(k)} = \alpha^{(k-1)}(t,x)x + \sigma^{(k-1)}(t,x)P_{\perp}^{(1)}, \tag{24}$$

where $\alpha^{(k-1)} = -\frac{1}{2}\nu\langle\nabla, U^{(k)}\rangle$ and $\sigma^{(k-1)}$ is a $(k-1)$-form with coefficients smooth functions of t.

From (20), we express $\langle U^{(k)}, P^{(1)}\rangle$ through $\langle\nabla, U^{(k)}\rangle S^{(2)}$ and substitute into (23). Then we obtain

$$\langle U^{(k)}, \nabla\rangle U^{(k)} = \langle\nabla, U^{(k)}\rangle U^{(k)}.$$

Simple computations show that this equation can be rewritten as

$$\det\left(\frac{\partial U^{(k)}}{\partial x}\right) = 0.$$

It follows that the vectors $\frac{\partial U_1^{(k)}}{\partial x}$ and $\frac{\partial U_2^{(k)}}{\partial x}$ are collinear, i.e., $\frac{\partial U_1^{(k)}}{\partial x} = \gamma_0 \frac{\partial U_2^{(k)}}{\partial x}$, where $\gamma_0(t)$ is a smooth function. By integrating these relations with respect to x and with regard for the fact that the $U^{(k)}$ are homogeneous polynomials in x, we readily obtain $U_2^{(k)} = U_1^{(k)}\gamma_0$.

Let us substitute this expression into (24) and multiply both sides of the resulting relation by $\mathbf{T}x$ in the sense of the inner product; this gives

$$U_1^{(k)}(x_2 - \gamma_0 x_1) = 2\sigma^{(k-1)}S^{(2)},$$

which contradicts the assumption that $U^{(k)}$ is not divisible by $S^{(2)}$, since $(x_2 - x_1\gamma_0) \neq 0$. This completes the proof of Lemma 10 as well as the part of Lemma 6 pertaining to equations (14) and (15).

The consideration of the case of systems (16), (17) is based on the similar ideas, but needs more sophisticated calculations; we omit them.

In conclusion let us note, that the corresponding system for $F = \sqrt{\tau}$ is

$$A + 2SB = 0, \quad C + 2SD + 2E = 0,$$

and, in contrast to the previous cases, it can be consistent if U and R have non-vanishing derivatives. This case is studied in [3, 4, 5, 8].

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